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TRANSIENT WAVE PROPAGATION NORMAL TO THE LAYERING OF A FINITE LAYERED MEDIUM†

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Abstract—Plane wave propagation in the direction normal to the layering of a periodically layered medium is studied. A period consists of two layers of homogeneous, isotropic, linear elastic or viscoelastic materials. The layered medium is of finite extent and hence consists of a finite number of layers. A theory is presented by which the layered medium is replaced by an "equivalent" linear homogeneous viscoelastic material such that the stress or the velocity in the latter and in the layered medium are identical at the centers of the alternate layers. Transient waves in the layered medium are then obtained by solving the transient waves in the "equivalent" homogeneous viscoelastic medium. Solutions at points other than the centers of the alternate layers are also presented. Numerical examples are given for transient waves in an elastic layered medium due to a step load applied at one of the boundary while the other boundary is fixed. Comparisons with the exact solutions by the ray theory show that the present theory can predict very satisfactorily transient waves in a finite layered medium.

1. INTRODUCTION

Most of the approximate theories for wave propagation in a layered medium focus on the determination of the dispersion relation or the frequency equation due to a harmonic oscillation[1–4], although some of the theories are able to predict the late-time asymptotic solution in a semi-infinite layered medium due to a step load applied at the boundary. For the latter, exact theories may be used to find the asymptotic solution and the wave-front solution[5–7].

To predict the transient response at points not necessarily far away from the impact end (where the asymptotic solution does not apply) and to points not necessarily near the wave-front, a new theory based on the analogy between the dynamic response of a semi-infinite layered medium and a semi-infinite homogeneous viscoelastic medium has been proposed recently by Ting and Mukunoki[8]. The fundamental idea is to characterize the layered medium by an "equivalent" homogeneous viscoelastic medium such that the dynamic response of the latter is identical to that of the layered medium at the centers of the alternate layers. Although the idea of modeling a composite by a viscoelastic medium is not new[9, 10], the "theory of viscoelastic analogy" introduced in [8] succeeds in correlating precisely the analogy between a layered medium and a homogeneous viscoelastic medium. Since wave propagation in a homogeneous linear viscoelastic medium can be solved easily by many known numerical schemes (see e.g. [11]), one can obtain the transient wave solution in a layered medium by solving the transient waves in the "equivalent" homogeneous viscoelastic medium.

The layered medium considered in [8] is of semi-infinite extent. In this paper we extend the theory to the case of a finite layered medium. First, we derive the general solution in the form of Laplace transform for waves propagating normal to the layerings of a finite layered medium. The general solution, which is applicable to any point in the finite layered medium, contains two arbitrary coefficients which can be determined from the boundary conditions of the finite layered medium. Next, we apply the general solution to certain points in the layered medium, namely, the centers of each layer. We show that the general solution at the centers of each layer is analogous to the general solution for waves propagating in a homogeneous viscoelastic medium. From this analogy we obtain the viscoelastic relaxation function of the "equivalent" homogeneous viscoelastic medium. Several analogies can be made depending on whether one is interested in the stress response or the velocity response in the layered medium. The analogies obtained here are more general than that presented in [8] and can be applied to the semi-infinite

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medium as well. In finding a means for determining the response at points other than the centers of the layers, we inadvertently obtain a characteristic relation in an integral form for one-dimensional waves in homogeneous viscoelastic media.

2. BASIC EQUATIONS

Consider a periodic layered medium as shown in Fig. 1 in which each period 2ω consists of two layers of homogeneous, isotropic, linear elastic or viscoelastic materials. The two different materials in the layers will be designated as material 1 and 2, respectively. Thus material 1 occupies layers $1, 3, 5, \ldots$ while material 2 occupies layers $2, 4, 6, \ldots$. The thicknesses of individual layers are denoted by $2h_i$ $(i = 1, 2)$ where the subscripts 1 and 2 refer to material 1 and 2, respectively. We will assume that the layered medium is initially at rest and occupies the region $0 \le x \le l$. We choose the central surface of layer 1 as $x = 0$ and the other boundary, $x = l$, is assumed to be at the central surface of layer N where N can be an even or odd integer. Hence,

$$l = (N - 1)\omega. \tag{1}$$

We will consider plane wave propagation in the direction x in which the only non-vanishing component of the displacement is in the x direction. We therefore have a one-dimensional wave propagation problem in which the equation of motion and the continuity of the displacement are given by

$$\frac{\partial \sigma_i}{\partial x} = \rho_i \dot{v}_i, \qquad (i = 1, 2) \tag{2}$$

$$\frac{\partial v_i}{\partial x} = \dot{\epsilon}_i, \qquad (i = 1, 2) \tag{3}$$

where a dot stands for differentiation with respect to the time t, and σ_i, ϵ_i, v_i, ρ_i $(i = 1, 2)$ are the normal stress, normal strain, particle velocity and mass density, respectively. Let $\lambda_i(t)$ and $\mu_i(t)$ be the relaxation functions of the materials. For elastic materials, $\lambda_i(t)$ and $\mu_i(t)$ are independent of t and are identified as Lamé constants. The stress–strain relation can be written in the form of Stieltjes convolution

$$\sigma_i(x, t) = \int_{0^-}^{t} g_i(t - t')\, \mathrm{d}\epsilon_i(x, t'), \tag{4}$$

$$g_i(t) = \lambda_i(t) + 2\mu_i(t), \tag{5}$$

where we have assumed that

$$\sigma_i(x, 0^-) = v_i(x, 0^-) = \epsilon_i(x, 0^-) = 0. \tag{6}$$

3. GENERAL SOLUTIONS

The general solution to eqns (2)–(6) can be obtained by the method of Laplace transform and by the use of the Floquet theory. We define the Laplace transform, $\bar{f}(p)$, of a function $f(t)$ by

$$\bar{f}(p) = \int_{0^-}^{t} f(t)\, \mathrm{e}^{-pt}\, \mathrm{d}t. \tag{7}$$

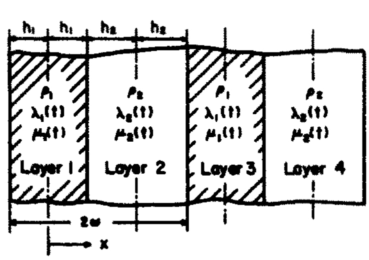

Fig. 1. Geometry of the layered medium.

After applying the Laplace transform to eqns (2)–(6), the general solution for the stress and the velocity in layers 1 and 2 can be written as

$$\bar{\sigma}_1(x,p) = \bar{A}_1 \cosh(k_1x) + \bar{B}_1 \sinh(k_1x) \tag{8a}$$

$$\bar{v}_1(x,p) = \frac{1}{m_1}\{\bar{A}_1 \sinh(k_1x) + \bar{B}_1 \cosh(k_1x)\} \tag{8b}$$

$$\bar{\sigma}_2(x,p) = \bar{A}_2 \cosh(k_2x - k_2\omega) + \bar{B}_2 \sinh(k_2x - k_2\omega) \tag{8c}$$

$$\bar{v}_2(x,p) = \frac{1}{m_2}\{\bar{A}_2 \sinh(k_2x - k_2\omega) + \bar{B}_2 \cosh(k_2x - k_2\omega)\} \tag{8d}$$

where

$$\left.\begin{aligned} \omega &= h_1 + h_2 \\ k_i &= \sqrt{(\rho_i p/\bar{g}_i)} \\ m_i &= \rho_i p/k_i = \sqrt{(\rho_i p \bar{g}_i)}. \end{aligned}\right\} \tag{9}$$

$\bar{A}_i$ and $\bar{B}_i$ $(i = 1, 2)$ are determined by the continuity condition at $x = h_1$

$$\begin{bmatrix} \bar{\sigma}_1 \\ \bar{v}_1 \end{bmatrix}(h_1, p) = \begin{bmatrix} \bar{\sigma}_2 \\ \bar{v}_2 \end{bmatrix}(h_1, p) \tag{10}$$

and the quasi-periodicity property of the solution together with the continuity condition at $x = 2\omega - h_1$

$$\begin{bmatrix} \bar{\sigma}_2 \\ \bar{v}_2 \end{bmatrix}(2\omega - h_1, p) = \begin{bmatrix} \bar{\sigma}_1 \\ \bar{v}_1 \end{bmatrix}(-h_1, p)\,e^{-2\omega\kappa} \tag{11}$$

where κ is the characteristic exponent[12]. Substitution of eqns (8) into eqns (10) and (11) leads to four homogeneous equations for $\bar{A}_i$ and $\bar{B}_i$. The requirement for a non-trivial solution results in the following equation for the characteristic exponent κ:

$$\cosh(2\omega\kappa) = \theta \cosh(2k_1h_1 + 2k_2h_2) - (\theta - 1)\cosh(2k_1h_1 - 2k_2h_2) \tag{12}$$

$$\theta = \frac{1}{4}\left(\frac{m_2}{m_1} + 2 + \frac{m_1}{m_2}\right). \tag{13}$$

Moreover, $\bar{A}_i$ and $\bar{B}_i$ are related by

$$\left.\begin{aligned} \frac{\bar{A}_2}{\bar{A}_1} &= p\bar{M}\,e^{-\omega\kappa}, & \frac{\bar{B}_2}{\bar{A}_1} &= -p\bar{R}m_2\,e^{-\omega\kappa}, \\ \frac{\bar{B}_1}{\bar{A}_1} &= -p\bar{L}_1, & \frac{\bar{B}_2}{\bar{A}_2} &= -p\bar{L}_2, \end{aligned}\right\} \tag{14}$$

where

$$\left.\begin{aligned} &\bar{L}_1 = m_1\bar{R}p\bar{M}, \qquad \bar{L}_2 = m_2\bar{R}/(p\bar{M}) \\ &p\bar{M} = \frac{m_1C_1C_2 + m_2S_1S_2}{m_1\cosh(\omega\kappa)} = \frac{m_2\cosh(\omega\kappa)}{m_2C_1C_2 + m_1S_1S_2} \\ &p\bar{R} = \frac{m_1C_1S_2 + m_2C_2S_1}{m_1m_2\sinh(\omega\kappa)} = \frac{\sinh(\omega\kappa)}{m_2C_1S_2 + m_1C_2S_1} \\ &C_i = \cosh(k_ih_i), \qquad S_i = \sinh(k_ih_i). \end{aligned}\right\} \tag{15}$$

Notice that if we interchange the subscripts 1 and 2, the expression for $p\bar{R}$ remains unchanged while $p\bar{M}$ becomes $(p\bar{M})^{-1}$. Therefore, we can obtain the Stieltjes inversion of $M(t)$ by sinply interchanging the subscripts 1 and 2 in the expression for $p\bar{M}$ and applying the Laplace inverse transform.

With eqn (14), the general solution in the layers 1 and 2 as expressed by eqn (8) can now be reduced to a solution containing only one coefficient, say $\bar{A}_1$. The solutions in other layers are obtained by the quasi-periodicity relation:

$$\begin{bmatrix} \bar{\sigma}_i \\ \bar{v}_i \end{bmatrix}(2n\omega + x, p) = \begin{bmatrix} \bar{\sigma}_i \\ \bar{v}_i \end{bmatrix}(x, p)\, e^{-2n\omega\kappa} \tag{16}$$

where n is an integer. Moreover, we see from eqn (12) that if κ is a characteristic exponent, so is $-\kappa$. Therefore, in addition to the general solution with $\bar{A}_1$ as the coefficient, we obtain the second general solution by changing the sign of κ. The coefficient of this second solution will be denoted by $\bar{A}_1'$. Consequently, the general solution for the stress and velocity at any point x in the layered medium can be written as, using eqns (8), (14), (16),

$$\begin{aligned} \bar{\sigma}_1(2n\omega + x_1, p) = {} & \bar{A}_1\{\cosh(k_1x_1) - p\bar{L}_1 \sinh(k_1x_1)\}\, e^{-2n\omega\kappa} \\ & + \bar{A}_1'\{\cosh(k_1x_1) + p\bar{L}_1 \sinh(k_1x_1)\}\, e^{2n\omega\kappa} \end{aligned} \tag{17a}$$

$$\begin{aligned} \bar{v}_1(2n\omega + x_1, p) = {} & \frac{\bar{A}_1}{m_1}\{\sinh(k_1x_1) - p\bar{L}_1 \cosh(k_1x_1)\}\, e^{-2n\omega\kappa} \\ & + \frac{\bar{A}_1'}{m_1}\{\sinh(k_1x_1) + p\bar{L}_1 \cosh(k_1x_1)\}\, e^{2n\omega\kappa} \end{aligned} \tag{17b}$$

$$\begin{aligned} \bar{\sigma}_2(2n\omega + \omega + x_2, p) = {} & \bar{A}_1 p\bar{M}\{\cosh(k_2x_2) - p\bar{L}_2 \sinh(k_2x_2)\}\, e^{-(2n+1)\omega\kappa} \\ & + \bar{A}_1' p\bar{M}\{\cosh(k_2x_2) + p\bar{L}_2 \sinh(k_2x_2)\}\, e^{(2n+1)\omega\kappa} \end{aligned} \tag{17c}$$

$$\begin{aligned} \bar{v}_2(2n\omega + \omega + x_2, p) = {} & \frac{\bar{A}_1}{m_2} p\bar{M}\{\sinh(k_2x_2) - p\bar{L}_2 \cosh(k_2x_2)\}\, e^{-(2n+1)\omega\kappa} \\ & + \frac{\bar{A}_1'}{m_2} p\bar{M}\{\sinh(k_2x_2) + p\bar{L}_2 \cosh(k_2x_2)\}\, e^{(2n+1)\omega\kappa} \end{aligned} \tag{17d}$$

where

$$-h_i \le x_i \le h_i, \qquad (i = 1, 2). \tag{18}$$

When proper values for n and x_1 (or x_2) are chosen, eqns (17) can be used to determine solution at any point in the layered medium. The two coefficients $\bar{A}_1$ and $\bar{A}_1'$ are determined from the boundary conditions at $x = 0$ and $x = l$.

In the next section we will show how one can obtain the solution at the centers of the layers by solving the wave propagation problem in an "equivalent" homogeneous viscoelastic medium. Having found the viscoelastic analogy for the solution at the centers of the layers, we then show how one can obtain the solution at points other than the centers of the layers in terms of the solution at the centers of the layers.

4. SOLUTION AT CENTERS OF LAYERS

The stress and velocity at the centers of the layers have specially simple forms. By letting $x_1 = x_2 = 0$ in eqn (17), we have

$$\left.\begin{aligned} \bar{\sigma}_1(2n\omega, p) &= \bar{A}_1\, e^{-2n\omega\kappa} + \bar{A}_1'\, e^{2n\omega\kappa} \\ \bar{v}_1(2n\omega, p) &= \frac{p\bar{L}_1}{m_1}(-\bar{A}_1\, e^{-2n\omega\kappa} + \bar{A}_1'\, e^{2n\omega\kappa}) \\ \bar{\sigma}_2(2n\omega + \omega, p) &= p\bar{M}(\bar{A}_1\, e^{-(2n+1)\omega\kappa} + \bar{A}_1'\, e^{(2n+1)\omega\kappa}) \\ \bar{v}_2(2n\omega + \omega, p) &= \frac{p\bar{L}_2}{m_2} p\bar{M}(-\bar{A}_1\, e^{-(2n+1)\omega\kappa} + \bar{A}_1'\, e^{(2n+1)\omega\kappa}). \end{aligned}\right\} \tag{19}$$

We now consider a homogeneous, isotropic, linear viscoelastic medium which occupies $0 \leq x \leq l$ and which is at rest at $t = 0^-$ and is subjected to certain prescribed boundary conditions at $x = 0$ and $x = l$. Let Φ, η and V be the normal stress, normal strain and particle velocity, respectively. Also, let ρ and G be the "equivalent" mass density and the "equivalent" relaxation function of this homogeneous viscoelastic material. The equation of motion, the continuity condition, the stress–strain relation and the initial conditions are

$$\left.\begin{aligned} \frac{\partial \Phi}{\partial x} &= \rho \dot{V} \\ \frac{\partial V}{\partial x} &= \dot{\eta} \\ \Phi(x, t) &= \int_{0^-}^{t} G(t - t')\, \mathrm{d}\eta(x, t') \\ \Phi(x, 0^-) &= V(x, 0^-) = \eta(x, 0^-) = 0. \end{aligned}\right\} \tag{20}$$

By applying the Laplace transform to eqns (20), the general solution for the stress and velocity will contain the exponential term

$$\exp\left(\pm \sqrt{(\rho p/\bar{G})}\, x\right). \tag{21}$$

In view of the exponential terms in eqns (19), we will define the "equivalent" relaxation function $G(t)$ by the relation

$$\kappa = \sqrt{(\rho p/\bar{G})}. \tag{22}$$

We will also define the "equivalent" mass density ρ by the average mass density in the layered medium[4, 8]:

$$\rho = (\rho_1 h_1 + \rho_2 h_2)/(h_1 + h_2). \tag{23}$$

With eqn (22), the general solution to eqn (20) can be written as

$$\bar{\Phi}(x, p) = \bar{a}\, \mathrm{e}^{-\kappa x} + \bar{a}'\, \mathrm{e}^{\kappa x} \tag{24a}$$

$$\bar{V}(x, p) = \frac{\kappa}{\rho p}\left(-\bar{a}\, \mathrm{e}^{-\kappa x} + \bar{a}'\, \mathrm{e}^{\kappa x}\right) \tag{24b}$$

where $\bar{a}$ and $\bar{a}'$ are arbitrary functions of p.

There are several ways to identify the analogy between eqns (19) and (24). If the stress in material 1 is of main interest, we may set

$$\bar{A}_1 = \bar{a}, \qquad \bar{A}_1' = \bar{a}' \tag{25}$$

we then have

$$\left.\begin{aligned} \bar{\sigma}_1(x, p) &= \bar{\Phi}(x, p) \\ \bar{v}_1(x, p) &= p\bar{J}_1 \bar{V}(x, p) \end{aligned}\right\} \quad \text{for} \quad x = 2n\omega \tag{26a}$$

and

$$\left.\begin{aligned} \bar{\sigma}_2(x, p) &= p\bar{M}\bar{\Phi}(x, p) \\ \bar{v}_2(x, p) &= p\bar{M}\{p\bar{J}_2 \bar{V}(x, p)\} \end{aligned}\right\} \quad \text{for} \quad x = (2n+1)\omega \tag{26b}$$

where

$$\bar{J}_i = \frac{\rho p \bar{L}_i}{\kappa m_i}, \qquad (i = 1, 2). \tag{27}$$

It should be pointed out that while $\bar{\Phi}$ and $\bar{V}$ as given by eqn (24) are defined for all x, eqns (26a) and (26b) apply only to $x = 2n\omega$ and $x = (2n+1)\omega$, respectively. By using the identity,

$$\bar{J}_1/\bar{J}_2 = (p\bar{M})^2 = m_2\bar{L}_1/(m_1\bar{L}_2) \tag{28}$$

the last of eqn (26b) can be written as

$$\bar{v}_2(x, p) = \frac{1}{p\bar{M}}\{p\bar{J}_1\bar{V}(x, p)\}, \qquad x = (2n+1)\omega. \tag{29}$$

With eqn (29), we rewrite eqns (26) in the following form:

$$\sigma_1(2n\omega, t) = \Phi(2n\omega, t) \tag{30a}$$

$$v_1(2n\omega, t) = V^*(2n\omega, t) \tag{30b}$$

$$\sigma_2(2n\omega + \omega, t) = \int_{0^-}^{t} M(t - t')\,\mathrm{d}\Phi(2n\omega + \omega, t') \tag{30c}$$

$$v_2(2n\omega + \omega, t) = \int_{0^-}^{t} M^{-1}(t - t')\,\mathrm{d}V^*(2n\omega + \omega, t') \tag{30d}$$

where

$$V^*(x, t) = \int_{0^-}^{t} J_1(t - t')\,\mathrm{d}V(x, t') \tag{30e}$$

and M^{-1} is the Stieltjes inverse of M. (See the discussion following eqn (15) regarding the Stieltjes inverse of M.) Thus the stress and velocity at the centers of the layers are related to the stress Φ and velocity V in the "equivalent" homogeneous medium. In particular, the stress at the centers of the odd layers, $\sigma_1(2\omega, t)$, is identical to the stress Φ in the "equivalent" homogeneous viscoelastic medium.

As an illustration for the theory of viscoelastic analogy, we consider an elastic layered medium which is fixed at the center of the 14th layer (i.e. $l = 13\omega$) and subjected to a unit step stress applied at $x = 0$. Since the 14th layer is occupied by material 2, we have the folllowing boundary conditions:

$$\sigma_1(0, t) = H(t), \qquad v_2(13\omega, t) = 0 \tag{31a}$$

where $H(t)$ is the Heaviside step function. In view of eqns (30), the corresponding boundary conditions for the "equivalent" viscoelastic medium are:

$$\Phi(0, t) = H(t), \qquad V(13\omega, t) = 0. \tag{31b}$$

We now replace the elastic layered medium by the "equivalent" homogeneous viscoelastic medium whose mass density ρ and the relaxation function $G(t)$ are given by eqns (23) and (22). Because of the complicated expression for κ as given by eqn (12), analytical inversion of the Laplace transform $\bar{G}(p)$ from eqn (22) does not appear feasible. We therefore resort to a numerical Laplace inversion of $\bar{G}(p)$[13]. The result is shown in Fig. 2 along with the physical parameters of the elastic layered medium used in the calculation. The physical parameters are taken from [4]. Unlike for most real viscoelastic materials, the relaxation function for the "equivalent" viscoelastic medium is not a monotonically decreasing function of t. This was also

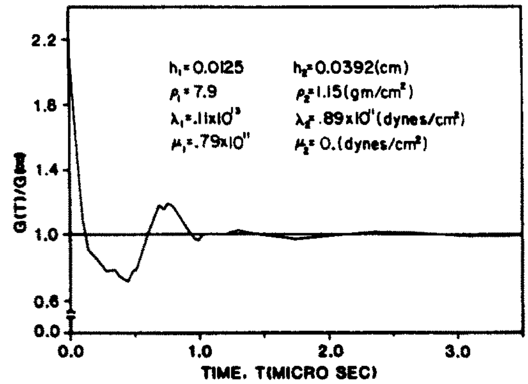

Fig. 2. "Equivalent" viscoelastic relaxation function $G(t)$.

predicted by Christensen[10] based on the dielectric theory. In [8] one can find a discussion on the behavior of $G(t)$ as $t \to 0$ and $t \to \infty$ as well as the value of $\dot{G}(t)$ at $t = 0$.

With $G(t)$ given by Fig. 2 and the boundary conditions given by eqn (31b), we integrate eqn (20) numerically by the method of characteristics[14] for the stress Φ and velocity V in the "equivalent" homogeneous viscoelastic medium. The stress and velocity at the centers of the layers in the layered medium are then determined by using the viscoelastic analogy eqns (30). In Fig. 3 we present $\Phi(4\omega, t)$ which is the stress history at the center of the 5th layer. For this example, the exact solution in the elastic layered medium using the ray theory can be obtained numerically by keeping track of every reflected and transmitted waves at the interfaces of the layers[8]. This exact solution is also shown in Fig. 3 for comparison. It is seen that the agreement is excellent.

In Fig. 3 we also show the solution obtained by the effective modulus theory[15]. With this theory, the elastic layered medium is replaced by a homogeneous elastic medium whose effective modulus g_{eff} is given by

$$\frac{1}{g_{\text{eff}}} = \left(\frac{h_1}{g_1} + \frac{h_2}{g_2}\right) \Big/ (h_1 + h_2) \tag{32}$$

and whose effective mass density is identical to the "equivalent" mass density defined in eqn (23).

In Figs. 4 and 5 we show, respectively, the velocity history at the center of the 5th layer and the stress history at the center of the 8th layer by using eqns (30b) and (30c). Since $\Phi(x, t)$ and $V(x, t)$ have already been determined, all we need is the functions $J_1(t)$ and $M(t)$ which are defined in eqns (27) and (15). $J_1(t)$ and $M(t)$ are obtained numerically by inverting their Laplace transforms. Again, the solutions by the ray theory and the effective modulus theory are also shown in the figures for comparison.

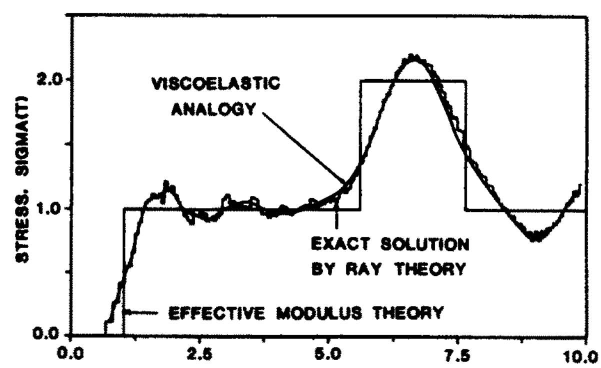

Fig. 3. Stress at the center of the 5th layer due to a unit step stress applied at the center of the first layer while the center of the 14th layer is fixed.

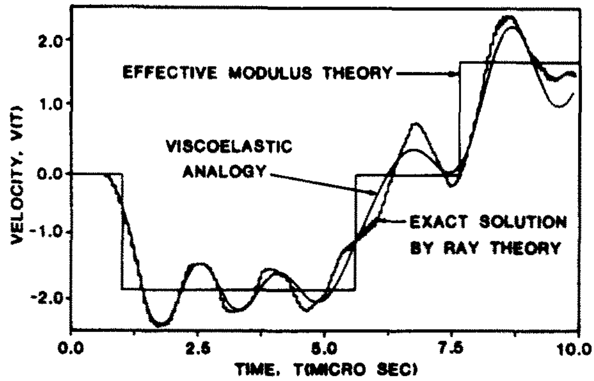

Fig. 4. Velocity at the center of the 5th layer due to a unit step stress applied at the center of the first layer while the center of the 14th layer is fixed.

The function J_i as well as functions M, R and L_i, $(i = 1, 2)$ defined in eqn (15) are called the "auxiliary" functions. Like $G(t)$, the auxiliary functions depend only on the physical properties and the geometrical layering of the layered medium. They are independent of the boundary conditions. For the viscoelastic analogy given by eqns (30), only the functions J_1, M and M^{-1} are needed. Of course, if $\sigma_1(2\omega, t)$ is the only quantity desired, no auxiliary functions are needed.

Before we study the solution at points other than the centers of the layers, we will discuss other forms of viscoelastic analogy in the next section.

5. OTHER FORMS OF VISCOELASTIC ANALOGY

In eqns (30) we present one form of viscoelastic analogy between eqns (19) and (24). The analogy, eqns (30), is convenient for the case when the stress in material 1 is of main interest because according to eqn (30a) σ_1 is identical to Φ. If the stress in material 2 is of main interest, then the analogy given by eqn (30c) requires a convolution integral with the auxiliary function $M(t)$.

There are of course other forms of viscoelastic analogy which would be more convenient for other situations. If the stress in material 2 is of main interest, one may set

$$\left.\begin{aligned} p\bar{M}\bar{A}_1 &= \bar{a} \\ p\bar{M}\bar{A}_1' &= \bar{a}' \end{aligned}\right\}. \tag{33}$$

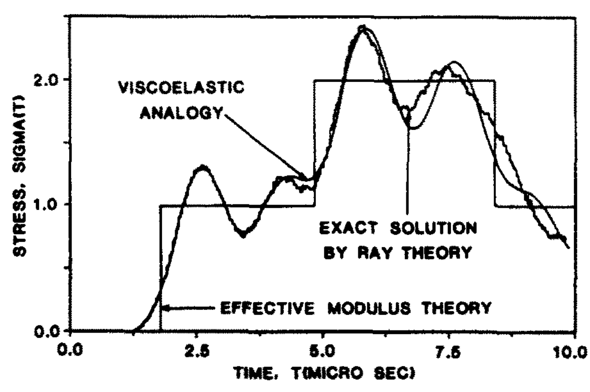

Fig. 5. Stress at the center of the 8th layer due to a unit step stress applied at the center of the first layer while the center of the 14th layer is fixed.

Then the analogy between eqns (19) and (24) can be written as

$$\left.\begin{aligned} \sigma_2(2n\omega+\omega,t) &= \Phi(2n\omega+\omega,t)\\ v_2(2n\omega+\omega,t) &= V^*(2n\omega+\omega,t)\\ \sigma_1(2n\omega,t) &= \int_{0^-}^{t} M^{-1}(t-t')\,\mathrm{d}\Phi(2n\omega,t')\\ v_1(2n\omega,t) &= \int_{0^-}^{t} M(t-t')\,\mathrm{d}V^*(2n\omega,t')\\ V^*(x,t) &= \int_{0^-}^{t} J_2(t-t')\,\mathrm{d}V(x,t'). \end{aligned}\right\} \tag{34}$$

With this analogy, the stress Φ in the "equivalent" viscoelastic medium is identical to σ_2 at $x=(2n+1)\omega$.

Likewise, if the velocity in material 1 is of main interest, the viscoelastic analogy can be written as

$$\left.\begin{aligned} v_1(2n\omega,t) &= V(2n\omega,t)\\ \sigma_1(2n\omega,t) &= \Phi^*(2n\omega,t)\\ v_2(2n\omega+\omega,t) &= \int_{0^-}^{t} M^{-1}(t-t')\,\mathrm{d}V(2n\omega+\omega,t')\\ \sigma_2(2n\omega+\omega,t) &= \int_{0^-}^{t} M(t-t')\,\mathrm{d}\Phi^*(2n\omega+\omega,t')\\ \Phi^*(x,t) &= \int_{0^-}^{t} J_1^{-1}(t-t')\,\mathrm{d}\Phi(x,t'). \end{aligned}\right\} \tag{35}$$

Finally, if the velocity in material 2 is of main interest, we can write

$$\left.\begin{aligned} v_2(2n\omega+\omega,t) &= V(2n\omega+\omega,t)\\ \sigma_2(2n\omega+\omega,t) &= \Phi^*(2n\omega+\omega,t)\\ v_1(2n\omega,t) &= \int_{0^-}^{t} M(t-t')\,\mathrm{d}V(2n\omega,t')\\ \sigma_1(2n\omega,t) &= \int_{0^-}^{t} M^{-1}(t-t')\,\mathrm{d}\Phi^*(2n\omega,t')\\ \Phi^*(x,t) &= \int_{0^-}^{t} J_2^{-1}(t-t')\,\mathrm{d}\Phi(x,t'). \end{aligned}\right\} \tag{36}$$

Sometimes the boundary conditions may influence the choice of a viscoelastic analogy. For example, suppose that one is interested in the stress in material 1 for the problem in which the velocity is prescribed at $x=0$ (i.e. $v_1(0,t)$ is known), and the other boundary $x=l$ is fixed. We could use either the analogy eqns (30) or the analogy eqns (35). If we use eqns (30), we obtain σ_1 directly from Φ but then we have to transform the boundary condition $v_1(0,t)$ to $V(0,t)$ by using eqns (30b) and (30e),

$$V(0,t)=\int_{0^-}^{t} J_1^{-1}(t-t')\,\mathrm{d}v_1(0,t') \tag{37}$$

before we solve for Φ and V in the "equivalent" homogeneous viscoelastic medium. If we use eqns (35), we can solve for Φ and V immediately since $V(0,t)=v_1(0,t)$, but to obtain σ_1 from Φ a convolution integral is required.

6. SOLUTION AT ARBITRARY POINTS

If we solve for $\bar{A}_1$ and $\bar{A}_1'$ from the first two equations of eqns (19) and substitute the results into eqns (17a) and (17b), we have

$$\bar{\sigma}_1(2n\omega + x_1, p) = \frac{1}{2}(e^{k_1x_1} + e^{-k_1x_1})\bar{\sigma}_1(2n\omega, p) + \frac{1}{2}(e^{k_1x_1} - e^{-k_1x_1})m_1\bar{v}_1(2n\omega, p) \quad (38a)$$

$$\bar{v}_1(2n\omega + x_1, p) = \frac{1}{2}(e^{k_1x_1} + e^{-k_1x_1})\bar{v}_1(2n\omega, p) + \frac{1}{2}(e^{k_1x_1} - e^{-k_1x_1})\frac{1}{m_1}\bar{\sigma}_1(2n\omega, p). \quad (38b)$$

Similar results can be obtained for $\sigma_2(2n\omega + \omega + x_2, p)$ and $v_2(2n\omega + \omega + x_2, p)$.

We will define the Laplace transform of the functions $D_1(x_1, t)$, $E_1(x_1, t)$ and $F_1(x_1, t)$ by

$$\left.\begin{aligned} \bar{D}_1(x_1, p) &= \frac{1}{p}e^{-k_1x_1} \\ \bar{E}_1(x_1, p) &= \frac{1}{pm_1}e^{-k_1x_1} \\ \bar{F}_1(x_1, p) &= \frac{m_1}{p}e^{-k_1x_1}. \end{aligned}\right\} \quad (39)$$

D_1, E_1 and F_1 have the following physical interpretation. Suppose that material 1 occupies the semi-infinite space $x \geq 0$ and is initially at rest. Then $D_1(x_1, t)$ and $E_1(x_1, t)$ are, respectively, the stress and velocity history at $x = x_1$ due to a unit step normal stress applied at $x = 0$. $F_1(x_1, t)$ is the stress history at $x = x_1$ due to a unit step velocity applied at $x = 0$. We now rewrite eqn (38a) as

$$\begin{aligned} \bar{\sigma}_1(2n\omega + x_1, p) = &\frac{1}{2}\{\bar{\sigma}_1(2n\omega, p)\,e^{px_1/c_{10}}\}p\{\bar{D}_1(-x_1, p)\,e^{-px_1/c_{10}}\} \\ &+\frac{1}{2}\{\bar{\sigma}_1(2n\omega, p)\,e^{-px_1/c_{10}}\}p\{\bar{D}_1(x_1, p)\,e^{px_1/c_{10}}\} \\ &+\frac{1}{2}\{\bar{v}_1(2n\omega, p)\,e^{px_1/c_{10}}\}p\{\bar{F}_1(-x_1, p)\,e^{-px_1/c_{10}}\} \\ &-\frac{1}{2}\{\bar{v}_1(2n\omega, p)\,e^{-px_1/c_{10}}\}p\{\bar{F}_1(x_1, p)\,e^{px_1/c_{10}}\} \end{aligned} \quad (40)$$

where

$$c_{i0} = \sqrt{(g_i(0)/\rho_i)}, \qquad (i = 1, 2) \quad (41)$$

and $D_1(-x_1, t)$ is obtained from $D_1(x_1, t)$ by analytically extrapolating the later from $x_1 > 0$ to $x_1 < 0$. Similar definition applies to $F_1(-x_1, t)$. Equation (40) can now be inverted as

$$\begin{aligned} \sigma_1(2n\omega + x_1, t) = &\frac{1}{2}\int_{0^-}^{t}\sigma_1\left(2n\omega, t + \frac{x_1}{c_{10}} - t'\right)dD_1\left(-x_1, t' - \frac{x_1}{c_{10}}\right) \\ &+\frac{1}{2}\int_{0^-}^{t}\sigma_1\left(2n\omega, t - \frac{x_1}{c_{10}} - t'\right)dD_1\left(x_1, t' + \frac{x_1}{c_{10}}\right) \\ &+\frac{1}{2}\int_{0^-}^{t}v_1\left(2n\omega, t + \frac{x_1}{c_{10}} - t'\right)dF_1\left(-x_1, t' - \frac{x_1}{c_{10}}\right) \\ &-\frac{1}{2}\int_{0^-}^{t}v_1\left(2n\omega, t - \frac{x_1}{c_{10}} - t'\right)dF_1\left(x_1, t' + \frac{x_1}{c_{10}}\right). \end{aligned} \quad (42)$$

Equation (42) can be written in a compact form if we observe that $\sigma_1(2n\omega, t)$ and $v_1(2n\omega, t)$ vanish for $t<0$, and $D_1(\pm x_1, t)$ and $F_1(\pm x_1, t)$ vanish for $t<\pm x_1/c_{10}$ and that $-h_1\le x_1\le h_1$ by eqn (18). We have

$$\sigma_1(2n\omega+x_1,t)=\frac{1}{2}\int_{(-h_1/c_{10})^-}^{t}\sigma_1(2n\omega,t-t')\,\mathrm{d}\{D_1(-x_1,t')+D_1(x_1,t')\}$$
$$+\frac{1}{2}\int_{(-h_1/c_{10})^-}^{t}v_1(2n\omega,t-t')\,\mathrm{d}\{F_1(-x_1,t')-F_1(x_1,t')\}. \tag{43a}$$

By a similar argument, we obtain from eqn (38b),

$$v_1(2n\omega+x_1,t)=\frac{1}{2}\int_{(-h_1/c_{10})^-}^{t}v_1(2n\omega,t-t')\,\mathrm{d}\{D_1(-x_1,t')+D_1(x_1,t')\}$$
$$+\frac{1}{2}\int_{(-h_1/c_{10})^-}^{t}\sigma_1(2n\omega,t-t')\,\mathrm{d}\{E_1(-x_1,t')-E_1(x_1,t')\}. \tag{43b}$$

When material 1 is elastic, m_1 is a constant and D_1, E_1 and F_1 are step functions:

$$\left.\begin{aligned}D_1(\pm x_1,t)&=H\left(t\mp\frac{x_1}{c_{10}}\right)\\ E_1(\pm x_1,t)&=\frac{1}{m_1}H\left(t\mp\frac{x_1}{c_{10}}\right)\\ F_1(\pm x_1,t)&=m_1H\left(t\mp\frac{x_1}{c_{10}}\right).\end{aligned}\right\} \tag{44}$$

Equations (43) then reduce to (when material 1 is elastic)

$$\sigma_1(2n\omega+x_1,t)=\frac{1}{2}\left\{\sigma_1\left(2n\omega,t+\frac{x_1}{c_{10}}\right)+\sigma_1\left(2n\omega,t-\frac{x_1}{c_{10}}\right)\right\}$$
$$+\frac{m_1}{2}\left\{v_1\left(2n\omega,t+\frac{x_1}{c_{10}}\right)-v_1\left(2n\omega,t-\frac{x_1}{c_{10}}\right)\right\} \tag{45a}$$
$$v_1(2n\omega+x_1,t)=\frac{1}{2}\left\{v_1\left(2n\omega,t+\frac{x_1}{c_{10}}\right)+v_1\left(2n\omega,t-\frac{x_1}{c_{10}}\right)\right\}$$
$$+\frac{1}{2m_1}\left\{\sigma_1\left(2n\omega,t+\frac{x_1}{c_{10}}\right)-\sigma_1\left(2n\omega,t-\frac{x_1}{c_{10}}\right)\right\}. \tag{45b}$$

This is nothing more than the familiar characteristic relation for one-dimensional waves in a homogeneous elastic medium. As such, eqn (43) may be regarded as the characteristic relation in an integral form for one-dimensional waves in a homogeneous viscoelastic medium.

After finding the stress and velocity at the center of the 5th layer in Figs. 3 and 4, we use eqn (45a) to find the stress history at the interface between the 4th and the 5th layers by letting $n=4$ and $x_1=-h_1$. The result is shown in Fig. 6 along with the exact solution by the ray theory.

7. DISCUSSION AND CONCLUDING REMARKS

Several analogies between the solutions of transient waves in a finite layered medium and in a finite homogeneous viscoelastic medium are established. The analogies between the solutions apply to the centers of the alternate layers in the layered medium. Solutions at points other than the centers of the alternate layers are obtained in terms of the solutions at the centers of the alternate layers through the use of convolution integrals with auxiliary functions introduced in the paper. The materials in the individual layer of the layered medium can be elastic or viscoelastic, although numerical examples are given for an elastic layered medium.

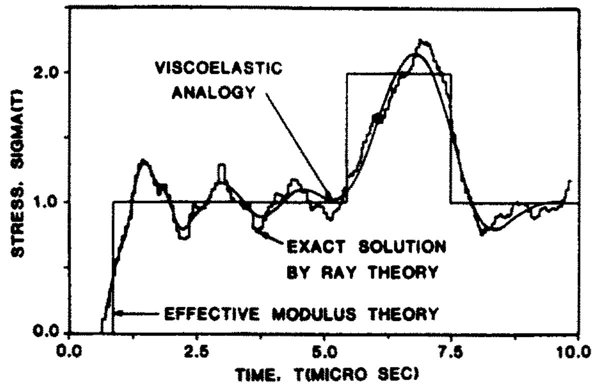

Fig. 6. Stress at the interface between the 4th and 5th layers due to a unit step stress applied at the center of the first layer while the center of the 14th layer is fixed.

The viscoelastic analogies derived here lead us to an exact expression $\bar{G}(p)$, which is the Laplace transform of the relaxation function $G(t)$ for the "equivalent" homogeneous viscoelastic medium. The relaxation function $G(t)$ is obtained by numerically inverting its Laplace transform using the method outlined in [13, 14]. The method used in [13, 14] tends to provide the value of $G(t)$ only at a finite number of t's which are close to $t = 0$. It provides poor information on $G(t)$ for large t. Fortunately, our relaxation function approaches rapidly to the equilibrium relaxation modulus G_∞ so that the inversion obtained by the method in [13, 14] is adequate for most cases. One could certainly obtain a better numerical inversion of a Laplace transform by using other tehcniques such as the fast Fourier transform [16].

Even though the relaxation function $G(t)$ obtained here is quite crude, we have reached an excellent agreement between the solutions by the viscoelastic analogy and by the exact ray theory. The differences in the solutions are caused mainly from the numerical Laplace inversion of the relaxation function $\bar{G}(p)$ and the auxiliary functions $\bar{M}(p)$ and $\bar{J}_1(p)$. For those solutions which require no auxiliary functions, the differences in the solutions are less noticeable. Some of the auxiliary functions, namely, M, R, L_1 and L_2 defined in eqn (15) can be determined exactly when the constituents of the layered medium are elastic. With the use of exact auxiliary functions, the differences between the solutions by the viscoelastic analogy and by the ray theory can be made smaller [14].

One might ask the relative advantages of the viscoelastic analogy over a direct numerical computation of the original layered problem. In a direct computation of waves in the original layered medium, the calculations are feasible when the boundary conditions are constant in time and the individual layers are elastic. Moreover, keeping track of every reflection and transmission of waves at the interfaces between the layers may soon exhaust the storage capacity of the computor, not to mention the computing time. The situation is particularly acute when there are a large number of layers involved as in a real composite. These shortcomings are not present in the theory of viscoelastic analogy. One possible shortcoming of the theory of viscoelastic analogy approach is the less accurate result for the solution at points other than the centers of the layers. This shortcoming is not serious in practical applications when the individual layers are very thin such that the solution at the center of a layer and at other points in the layer are practically the same.

In connection with the present work on transient waves in a finite layered medium, we would like to point out that harmonic waves in a finite layered medium has been considered by Herrmann, Beaupre and Auld [17]. In contrast to the normal stress waves studied here, the harmonic waves considered in [17] are horizontally polarized shear waves. However, one can see from the analyses presented here that if we replace the normal stress and the normal displacement by the shear stress and the transverse displacement, respectively, the analyses presented here apply to the transient shear waves in finite layered media as well.

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